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Abelian vortices on nodal and cuspidal curves

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Abstract: We compute the Euler characteristics of the moduli spaces of abelian vortices on curves with nodal and cuspidal singularities. This generalizes our previous work where only nodes were taken into account. The result we obtain is again consistent with the expected reconciliation between the vortex picture of $D2$-$D0$ branes and the proposal by Gopakumar and Vafa.

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Contents

1 Introduction

Let C be a nonsingular complex projective curve of genus g. The moduli space of abelian vortices on C is well-known to be described by the d-fold symmetric product $C^{(d)}$ where d is the amount of magnetic flux. Its Euler characteristic $\chi(C^{(d)})$ can be computed via the generating function [1]

$$\sum_{d=0}^{\infty} \chi(C^{(d)}) y^{d+1-g} = (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{2g-2} \tag{1.1}$$

where $0 < |y| < 1$ is assumed. In view of the existence of the Abel-Jacobi map from $C^{(d)}$ to the Jacobian $J(C)$ it is not unreasonable to expect a close relation between the two. The total complex cohomology ring $H^*(J(C))$ is an $\mathfrak{sl}_2$ module under the Lefschetz $\mathfrak{sl}_2$ action. If we denote the Cartan generator of the $\mathfrak{sl}_2$ by H, we have

$$\mathrm{Tr}_{H^*(J(C))}(-1)^{\mathsf{H}} y^{\mathsf{H}} = (-1)^g (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{2g}. \tag{1.2}$$

Then, we observe that (1.1) and (1.2) coincide up to a simple factor $(-1)^g (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2$.

Abelian vortices on a curve are expected to describe the bound system of a single $D2$-brane coupled to $D0$-branes. In this context, the above observation, though it may look accidental, is crucial for the reconciliation between the vortex picture of $D2$-$D0$ branes [2] and the proposal by Gopakumar and Vafa [3]. Recall that the latter is motivated by an effective theory consideration[1] and tries to interpret the Lefschetz $\mathfrak{sl}_2$ action on the cohomologies of the Jacobian as the *half* of space-time Lorentz symmetry. In many interesting and important cases though, the curve around which the $D2$-brane is wrapping can be singular and a priori one is not sure if the same kind of simple relation holds. Nevertheless, such a relation seems to be required if one believes in the compatibility of the two pictures. In [4], we studied this issue when the singularities of the curve are nodes and found that the two expressions are again simply related as in the nonsingular case. In this short note, we modestly extend this result by additionally allowing cusps on the curve. See [4] for more on the motivation behind the present work and the background materials.

The main computation for abelian vortices on nodal and cuspidal curves is given in section 2. We compare this result with the Gopakumar-Vafa type expression for the compactified Jacobians in section 3.

[1] In the sense of string compactification.

2 Abelian vortices on nodal and cuspidal curves

Let C be an integral complex projective curve of arithmetic genus g having a nodes and b cusps as its only singularities. We denote by $C^{[d]}$ the Hilbert scheme of zero-dimensional subschemes of length d on C. One may regard $C^{[d]}$ as the moduli space of vortices on C. Then our claim is that

$$\begin{aligned}&\sum_{d=0}^{\infty}\chi(C^{[d]})y^{d+1-g}\\ &\quad=(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^{2g-2}\left(1+\frac{1}{(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^2}\right)^a\left(1+\frac{2}{(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^2}\right)^b\end{aligned} \tag{2.1}$$

for $0<|y|<1$.

In order to prove this, we first gather relevant materials on local punctual Hilbert schemes at singularities. The local punctual Hilbert scheme $\mathrm{Hilb}^{\ell}_{\propto}$ at a node parametrizes ideals of colength ℓ in $\mathbb{C}[\![\mathtt{x},\mathtt{y}]\!]/(\mathtt{xy})$. If $\ell>1$, such ideals are given by [5]

$$\begin{aligned}I^{\ell}_i(u_i)&=(\mathtt{y}^i+u_i\mathtt{x}^{\ell-i}),\quad (u_i\in\mathbb{C}^{\times}, i=1,\dots,\ell-1),\\ Q^{\ell}_i&=(\mathtt{x}^{\ell-i+1},\mathtt{y}^i),\quad (i=1,\dots,\ell)\end{aligned} \tag{2.2}$$

with the relations $\lim_{u_i\to 0}I^{\ell}_i(u_i)=Q^{\ell}_i$ and $\lim_{u_i\to\infty}I^{\ell}_i(u_i)=Q^{\ell}_{i+1}$. Hence $\mathrm{Hilb}^{\ell}_{\propto}$ with $\ell>1$ is a chain of $\ell-1$ rational curves configured as [5]:

The only colength one ideal is $Q^1_1=(\mathtt{x},\mathtt{y})$. Hence $\mathrm{Hilb}^1_{\propto}$ is a point.

The local punctual Hilbert scheme $\mathrm{Hilb}^{\ell}_{\prec}$ at a cusp parametrizes ideals of colength ℓ in $\mathbb{C}[\![\mathtt{t}^2,\mathtt{t}^3]\!](\cong\mathbb{C}[\![\mathtt{x},\mathtt{y}]\!]/(\mathtt{y}^2-\mathtt{x}^3))$. If $\ell>1$, such ideals are given by [6, 7]

$$\begin{aligned}I^{\ell}(u)&=(\mathtt{t}^{\ell}+u\mathtt{t}^{\ell+1}),\quad (u\in\mathbb{C}),\\ Q^{\ell}&=(\mathtt{t}^{\ell+1},\mathtt{t}^{\ell+2})\end{aligned} \tag{2.3}$$

with the relation $\lim_{u\to\infty}I^{\ell}(u)=Q^{\ell}$. Hence $\mathrm{Hilb}^{\ell}_{\prec}\cong\mathbb{P}^1$ if $\ell>1$. The only colength one ideal is $Q^1=(\mathtt{t}^2,\mathtt{t}^3)$. Thus $\mathrm{Hilb}^1_{\prec}$ is a point.

Recall that a partition is a sequence $\lambda=(\lambda_1,\lambda_2,\dots)$ of non-negative integers in non-increasing order and containing only finitely many non-zero terms. We say that λ is a partition of d if $|\lambda|:=\sum\lambda_i=d$. When λ is a partition of d, we use an alternative notation $\lambda=(1^{\delta_1}2^{\delta_2}\cdots d^{\delta_d})$ where $\delta_\ell=\#\{i\mid\lambda_i=\ell\}$ so that $\sum_{\ell=1}^{d}\ell\delta_\ell=d$.

Let $\mathcal{A}$ be the set of nodes on C and $\mathcal{B}$ that of cusps on C. The argument in [4] can be readily extended in the present case and we obtain

$$\begin{aligned}\chi(C^{[d]}) = \sum_{\substack{\lambda=(1^{\delta_1}\cdots d^{\delta_d})\\|\lambda|=d}} \sum_{\substack{\mathcal{A}=\sqcup_{\ell=1}^{d}\mathcal{A}_\ell\\ \mathcal{B}=\sqcup_{\ell=1}^{d}\mathcal{B}_\ell\\ \#\mathcal{A}=a,\,\#\mathcal{B}=b\\ \#\mathcal{A}_\ell+\#\mathcal{B}_\ell=\delta_\ell\ (\ell\geq 2)}} &\chi\left(\left(C\setminus\sqcup_{\ell=2}^{d}(\mathcal{A}_\ell\sqcup\mathcal{B}_\ell)\right)^{(\delta_1)}\right)\\ &\times\prod_{\ell=2}^{d}\left[\left(\chi(\mathrm{Hilb}^{\ell}_{\propto})-1\right)^{\#\mathcal{A}_\ell}\left(\chi(\mathrm{Hilb}^{\ell}_{\prec})-1\right)^{\#\mathcal{B}_\ell}\right].\end{aligned} \tag{2.4}$$

The explicit descriptions of $\mathrm{Hilb}^{\ell}_{\propto}$ and $\mathrm{Hilb}^{\ell}_{\prec}$ in the above imply that $\chi(\mathrm{Hilb}^{\ell}_{\propto}) = 2(\ell-1)-(\ell-2)=\ell$ and $\chi(\mathrm{Hilb}^{\ell}_{\prec})=2$ for $\ell>1$. Moreover,

$$\chi\left(\left(C\setminus\sqcup_{\ell=2}^{d}(\mathcal{A}_\ell\sqcup\mathcal{B}_\ell)\right)^{(\delta_1)}\right)=\binom{\delta_1-1+\chi\left(C\setminus\sqcup_{\ell=2}^{d}(\mathcal{A}_\ell\sqcup\mathcal{B}_\ell)\right)}{\delta_1}. \tag{2.5}$$

Hence, by setting $a_\ell=\#\mathcal{A}_\ell$ and $b_\ell=\#\mathcal{B}_\ell$, we see that

$$\begin{aligned}\chi(C^{[d]}) = \sum_{\substack{\lambda=(1^{\delta_1}\cdots d^{\delta_d}),\,|\lambda|=d,\\ \sum_{\ell\geq 2}a_\ell\leq a,\\ \sum_{\ell\geq 2}b_\ell\leq b,\\ a_\ell+b_\ell=\delta_\ell\ (\ell\geq 2)}} &\binom{a}{a-\sum_{\ell\geq 2}a_\ell,a_2,\ldots,a_d}\binom{b}{b-\sum_{\ell\geq 2}b_\ell,b_2,\ldots,b_d}\\ &\times\binom{\delta_1-1+\chi(C)-\sum_{\ell\geq 2}(a_\ell+b_\ell)}{\delta_1}\prod_{\ell=2}^{d}(\ell-1)^{a_\ell}.\end{aligned} \tag{2.6}$$

Now let us switch from the sum over partitions λ to that over a_ℓ's and b_ℓ's. Then,

$$\begin{aligned}\chi(C^{[d]}) = \sum_{\substack{a_2\geq 0,\ldots,a_d\geq 0\\ b_2\geq 0,\ldots,b_d\geq 0\\ \sum_{\ell\geq 2}\ell(a_\ell+b_\ell)\leq d\\ \sum_{\ell\geq 2}a_\ell\leq a\\ \sum_{\ell\geq 2}b_\ell\leq b}} &\binom{a}{\sum_{\ell\geq 2}a_\ell}\binom{\sum_{\ell\geq 2}a_\ell}{a_2,\ldots,a_d}\binom{b}{\sum_{\ell\geq 2}b_\ell}\binom{\sum_{\ell\geq 2}b_\ell}{b_2,\ldots,b_d}\\ &\times\binom{d-\sum_{\ell\geq 2}(\ell+1)(a_\ell+b_\ell)-1+\chi(C)}{d-\sum_{\ell\geq 2}\ell(a_\ell+b_\ell)}\prod_{\ell=2}^{d}(\ell-1)^{a_\ell}\,.\end{aligned} \tag{2.7}$$

Consequently, the generating function becomes

$$
\begin{aligned}
&\sum_{d=0}^{\infty}\chi(C^{[d]})y^d \\
&=\sum_{j=0}^{a}\sum_{\substack{a_2\geq 0,a_3\geq 0,\ldots\\ j=\sum_{\ell\geq 2}a_\ell}}\sum_{k=0}^{b}\sum_{\substack{b_2\geq 0,b_3\geq 0,\ldots\\ k=\sum_{\ell\geq 2}b_\ell}}\binom{a}{j}\binom{j}{a_2,a_3,\ldots}\binom{b}{k}\binom{k}{b_2,b_3,\ldots} \\
&\qquad\times\left(\prod_{\ell\geq 2}(\ell-1)^{a_\ell}\right)y^{\sum_{\ell\geq 2}\ell(a_\ell+b_\ell)}(1-y)^{\sum_{\ell\geq 2}(a_\ell+b_\ell)-\chi(C)} \\
&=(1-y)^{-\chi(C)}\sum_{j=0}^{a}\binom{a}{j}\sum_{\substack{a_2\geq 0,a_3\geq 0,\ldots\\ j=\sum_{\ell\geq 2}a_\ell}}\binom{j}{a_2,a_3,\ldots}\prod_{\ell\geq 2}\left\{(\ell-1)y^\ell(1-y)\right\}^{a_\ell} \\
&\qquad\times\sum_{k=0}^{b}\binom{b}{k}\sum_{\substack{b_2\geq 0,b_3\geq 0,\ldots\\ k=\sum_{m\geq 2}b_m}}\binom{k}{b_2,b_3,\ldots}\prod_{m\geq 2}\left\{y^m(1-y)\right\}^{b_m}
\end{aligned}
\tag{2.8}
$$

where we have used the binomial theorem in the first step. The multinomial theorem further simplifies the last expression as

$$
(1-y)^{-\chi(C)}\sum_{j=0}^{a}\binom{a}{j}\left(\sum_{\ell\geq 2}(\ell-1)y^\ell(1-y)\right)^j\sum_{k=0}^{b}\binom{b}{k}\left(\sum_{m\geq 2}y^m(1-y)\right)^k. \tag{2.9}
$$

Hence, by summing over ℓ and m we obtain that

$$
\sum_{d=0}^{\infty}\chi(C^{[d]})y^d=(1-y)^{-\chi(C)}\sum_{j=0}^{a}\binom{a}{j}\left(\frac{y^2}{1-y}\right)^j\sum_{k=0}^{b}\binom{b}{k}y^{2k}. \tag{2.10}
$$

Finally, the sums over j and k can be done by the binomial theorem:

$$
\sum_{d=0}^{\infty}\chi(C^{[d]})y^d=(1-y)^{-\chi(C)}\left(1+\frac{y^2}{1-y}\right)^a(1+y^2)^b. \tag{2.11}
$$

By using $\chi(C)=2-2g+a+2b$ one immediately recognizes that this is equivalent to (2.1).

3 Reconciliation with the Gopakumar-Vafa picture

Let $\nu:\tilde{C}\to C$ be the normalization. The generalized Jacobian $J(C)$ fits into an exact sequence of abelian algebraic groups

$$
1\to(\mathbb{G}_m)^a\times(\mathbb{G}_a)^b\to J(C)\xrightarrow{\nu^*}J(\tilde{C})\to 1 \tag{3.1}
$$

where $\mathbb{G}_m\cong\mathbb{C}^\times$ is the multiplicative group, $\mathbb{G}_a\cong\mathbb{C}$ is the additive group, 1 is the trivial group, and $J(\tilde{C})$ is the Jacobian of $\tilde{C}$. Thus to obtain the compactified Jacobian $\bar{J}(C)$

from $J(C)$ one needs appropriate compactifications of $\mathbb{G}_m$ and $\mathbb{G}_a$. Let $R_\propto$ be a rational curve with a node, $R_\prec$ a rational curve with a cusp. We know that the nonsingular parts of $R_\propto$ and $R_\prec$ are respectively isomorphic to $\mathbb{G}_m$ and $\mathbb{G}_a$ [8]. Hence $R_\propto$ and $R_\prec$ can be regarded as such compactifications.

To compare our result with the proposal by Gopakumar and Vafa [3] we need to know the "Lefschetz $\mathfrak{sl}_2$ action" on $H^*(\bar{J}(C))$. At this stage one might worry about the feasibility of this since the so-called "Kähler package" does not necessarily hold for the usual cohomologies of singular varieties. However, in the present case we may evade this obstacle by using the following argument. The curve $R_\propto$ is obtained by shrinking one of the two generators of $H_1(E)$ of an elliptic curve E. Similarly, $R_\prec$ is obtained by shrinking both of the two generators of $H_1(E)$. So, although $R_\propto$ and $R_\prec$ are singular, $H^*(R_\propto)$ and $H^*(R_\prec)$ may still be regarded as the $\mathfrak{sl}_2$ modules obtained by deleting respectively one spin 0 and two spin 0 representations from the $\mathfrak{sl}_2$ module $H^*(E)$. With this interpretation in mind, we have

$$\begin{aligned}\mathrm{Tr}_{H^*(R_\propto)}(-1)^{\mathsf{H}}y^{\mathsf{H}} &= -(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^2-1,\\ \mathrm{Tr}_{H^*(R_\prec)}(-1)^{\mathsf{H}}y^{\mathsf{H}} &= -(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^2-2.\end{aligned} \tag{3.2}$$

(Recall that the arithmetic genera of $R_\propto$, $R_\prec$ and E are all equal to one.) Since the genus of $\tilde{C}$ is $g-a-b$, it follows from (3.1) and (3.2) that

$$\begin{aligned}&\mathrm{Tr}_{H^*(\bar{J}(C))}(-1)^{\mathsf{H}}y^{\mathsf{H}}\\ &\quad=(-1)^g(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^{2(g-a-b)}\left\{(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^2+1\right\}^a\left\{(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^2+2\right\}^b.\end{aligned} \tag{3.3}$$

Hence we conclude that the expected relation indeed holds:

$$(-1)^g\sum_{d=0}^{\infty}\chi(C^{[d]})y^{d+1-g}=\frac{\mathrm{Tr}_{H^*(\bar{J}(C))}(-1)^{\mathsf{H}}y^{\mathsf{H}}}{(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^2}\,. \tag{3.4}$$

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Acknowledgments

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